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Branes, Geometry and $N = 1$ Duality with Product Gauge Groups of SO and Sp

CHANGHYUN AHN^{a,*}, KYUNGHO OH^{b,†} AND RADU TATAR^{c,‡}

^a APCTP, 207-43 Cheongryangri-dong, Dongdaemun-gu, Seoul 130-012, Korea and
 Dept. of Physics, Seoul National University, Seoul 151-742, Korea

^b Dept. of Mathematics, University of Missouri-St. Louis, St. Louis, Missouri 63121, USA

^c Dept. of Physics, University of Miami, Coral Gables, Florida 33146, USA

ABSTRACT

We study $N = 1$ dualities in four dimensional supersymmetric gauge theories as the worldvolume theory of D4 branes with one compact direction in type IIA string theory. We generalize the previous work for $SO(N_{c1}) \times Sp(N_{c2})$ with the superpotential $W = \text{Tr}X^4$ to the case of $W = \text{Tr}X^{4(k+1)}$ in terms of brane configuration. We conjecture that the *new* dualities for the product gauge groups of $SO(N_{c1}) \times Sp(N_{c2}) \times SO(N_{c3})$, $SO(N_{c1}) \times Sp(N_{c2}) \times SO(N_{c3}) \times Sp(N_{c4})$ and higher multiple product gauge groups can be obtained by reversing the ordering of NS5 branes and D6 branes while preserving the linking numbers. We also describe the above dualities in terms of wrapping D6 branes around 3 cycles of Calabi-Yau threefolds in type IIA string theory. The theory with adjoint matter can be regarded as taking multiple copies of NS5 brane in the configuration of brane or geometric approaches.

* chahn@spin.snu.ac.kr

† oh@arch.umsl.edu

‡tatar@phyvax.ir.miami.edu

1 Introduction

In the recent years it has become obvious that Dirichlet(D) branes provide an important tool for studying gauge theories in various dimensions. There are several approaches to this subject. One is to consider the compactification of F-theory on elliptic Calabi-Yau fourfolds from 12 dimensions which gives $N = 1$ pure supersymmetric $SU(N_c)$ gauge theories in four dimensions [1]. In [2], by studying a configuration with D3 branes and D7 branes, it has been shown that the local string model gives rise to $SU(N_c)$ Yang-Mills theory with matter in the fundamental representation.

The second approach was initiated by the crucial work of Hanany and Witten [3]. They have studied theories with $N = 4$ supersymmetry in 3 dimensions, obtaining a nice realization of the mirror symmetry. By considering type IIB string theory, they took a configuration which preserves 1/4 of the supersymmetry, i.e., $N = 2$ theory in 4 dimensions or $N = 4$ theory in 3 dimensions. Their configurations consisted of *parallel* Neveu-Schwarz(NS)5 branes with D3 branes suspended between them and D5 branes placed between them. A new aspect of brane dynamics was so called “Hanany-Witten effect”, leading to the creation of a D3 brane whenever a D5 brane and a NS5 brane are passing through each other. The original explanation was due to the conservation of the linking number defined as a total magnetic charge for the gauge field coupled with the worldvolume of the both types of NS and D branes. Another explanation of this effect was given in [4] using anomaly flow argument for two mutually orthogonal D4 branes passing through each other where a fundamental string is created. By a chain of T-dualities and S-dualities this reduces to exactly the original “Hanany-Witten effect”. Other explanations along the line of [3] were given in [5, 6, 7].

From Hanany and Witten’s work, the passage to $N = 1$ theories in 4 dimensions was taken in the important paper of Elizur, Giveon and Kutasov [8]. We call this EGK approach. They considered a configuration which preserves 1/8 of the supersymmetry, i.e., $N = 1$ theories in 4 dimensions. They considered type IIA string theory with *perpendicular* NS5 branes with D4 branes suspended between them and D6 branes placed between them. The D4 branes are finite in one direction and the worldvolume dynamics describes at long distances $N = 1$ supersymmetric field theories in 4 dimensions. The configuration of a D4 brane gives the gauge group while the D6 branes which are needed for matter give the global flavor group. The D4 brane worldvolume gauge coupling is related to the finite distance between the two NS5 branes. That is, the distance between two NS5 branes in compact direction determines the gauge coupling. Using this configuration, they were able to describe and check the Seiberg’s dualities for $N = 1$ supersymmetric gauge theory with $SU(N_c)$ gauge group with N_f flavors in the fundamental representation [9]. They changed the positions of the two NS5 branes in a smooth way, without breaking the supersymmetry and connected the Higgs phases of the two theo-

ries(electric theory and its magnetic dual theory). This method sparked an intense work in obtaining many other dualities for $N = 1$ theories [10, 11, 12, 13, 14, 15, 16, 17, 18]. Especially interesting are the papers of [10, 13] where the results of [19, 20] were rederived from brane configuration for the gauge groups $SO(N_c)$ and $Sp(N_c)$. They introduced supplementary orientifold O4 and O6 planes to construct orthogonal and symplectic gauge groups. In this case the NS5 branes have to pass over each other and some strong coupling phenomena have to be considered. For example, for the case of $SO(N_c)$ gauge group, they consist of the appearance of supplementary D4 branes or of some D4 branes which are forced to be stuck at the orientifold O4 plane in order to obtain a smooth transition.

The third approach was initiated in the important work of Ooguri and Vafa [21], using the results of [22, 23]. We call this OV approach. Instead of working in flat spacetime, they considered the compactification of IIA string theory on a double elliptically fibered CY threefold. They wrapped D6 branes around three cycles of CY threefold filling four dimensional spacetime. The transition between a theory and its magnetic dual appears when a change in the moduli space of CY threefold occurs. Their result was generalized in [24, 25, 26] to various other models which reproduce field theory results studied previously in [27, 28, 29, 30, 31, 32].

Recently, it has become evident that many important results can be obtained from strongly coupled type IIA string theory which is given by M theory. Initiated by the work of Witten [33], this provides an important way to describe strongly coupled gauge theories. Starting with a theory of M theory 5 branes in 11 dimensions, a dimensional reduction gives D4 branes (which is an M theory 5 brane wrapped over S^1) and NS5 branes (which is an M theory 5 brane on $\mathbf{R}^{10} \times S^1$) in 10 dimensions. To obtain D6 branes, one has to use a multiple Taub-NUT space whose metric is complete and smooth. The beta function receives a geometrical interpretation. Results of $N = 2$ gauge theories in 4 dimensions with gauge group $SU(N_c)$ appear naturally in the context of M theory. This approach was extended to other gauge groups in [34, 35, 36, 37] and to other dimensions [38, 39, 40]. Very recently, using a flow from $N = 2$ to $N = 1$ supersymmetric theories, non-perturbative results obtained in field theories were reproduced by using brane configurations in [41, 42]. Other important results are also obtained in [43, 44, 45].

In this paper we extend the results of [14, 26] to the case of multiple products of SO and Sp gauge groups both in EGK and OV approaches. As in [26], the condition to be satisfied in the dual theory is that the flavor groups are given only by D6 branes wrapped on cycles between *pairs of singularity points*, which are points where, in the T-dual picture, the NS5 branes which appear after the T-duality are parallel. In brane configuration picture, as in [11] this requires semi-infinite D4 branes also in our construction. The duality are also deduced and checked in field theory.

In section 2, we review the result of [14] and rederive it also from OV approach and generalize to any value of k and for adjoint matter. In section 3, we extend our results to $SO(N_{c1}) \times Sp(N_{c2}) \times SO(N_{c3})$ case for any value of k which appears in the superpotential and for adjoint matter. In section 4, we consider the case of a 4-tuple product gauge group. In section 5, we generalize to the case of n -tuple product gauge group. Finally in section 6, we conclude our results and comment on the outlook in the future direction.

2 Duality for $SO(N_{c1}) \times Sp(N_{c2})$

The brane configuration we study contains three kinds of branes in type IIA string theory: NS5 brane with worldvolume $(x^0, x^1, x^2, x^3, x^4, x^5)$, D6 brane with worldvolume $(x^0, x^1, x^2, x^3, x^7, x^8, x^9)$ and D4 brane with worldvolume $(x^0, x^1, x^2, x^3, x^6)$ where the x^6 direction is a finite interval. We will consider the case of an O4 orientifold in the EGK approach which is parallel to the D4 brane in order to preserve supersymmetry and is not of finite extent in x^6 direction. The D4 brane is the only brane which is not intersected by O4 orientifold. The orientifold gives a spacetime reflection as $(x^4, x^5, x^7, x^8, x^9) \rightarrow (-x^4, -x^5, -x^7, -x^8, -x^9)$ which are reflections on noncompact directions, in addition to the gauging of worldsheet parity Ω .

On the directions where the orientifold is a point, any object which is extended along them will have a mirror. That is, the NS5 branes have a mirror in (x^4, x^5) directions and D6 branes have one in (x^7, x^8, x^9) directions. These objects and their mirrors enter only once. It would be overcounting to treat an object and its reflection as separate physical objects. Therefore we should have the factors of one half in the counting of the number of physical objects later. Another important aspect of the orientifold is its charge, given by the charge of $H^{(6)} = dA^{(5)}$ coming from Ramond Ramond(RR) sector, which is related to the sign of Ω^2 . This is because the charge of the orientifold is directly connected to the charge of D branes. In the natural normalization, where the D4 brane carries one unit of this charge, the charge of the O4 plane is ± 1 , for $\Omega^2 = \mp 1$ in the D4 brane sector. In this section we will try to understand the field theory results for $N = 1$ dualities [46] in the context of EGK approach first and OV approach later on.

2.1 The superpotential $\mathbf{W} = \text{Tr } X^4$ case

Let us consider the electric theory for the simplest case in the sense that each of the three kinds of branes has its single copy. The mechanism for duality in the EGK approach is similar that of [11] except that we have to consider the orientifold reflection and

charges. In OV approach, the difference between this section and [21] is that we have to twist the doubly elliptic fibration due to the presence of the product gauge groups with flavors. We deal with $SO(N_{c1}) \times Sp(N_{c2})$ gauge group with $2N_{f1}(2N_{f2})$ flavors in the vector(fundamental) representation of $SO(N_{c1})(Sp(N_{c2}))$ group[§].

In the EGK configuration of [14], this corresponds to three NS5 branes where we need NS5 branes at arbitrary angles in (x^4, x^5, x^8, x^9) directions so that any two of the 5 branes are not parallel as discussed in [15]. We label them by A 5, B 5 and C 5 from left to right on the compact x^6 direction. There exist N_{c1} D4 branes stretched between A 5 brane and B 5 brane which is connected to C 5 brane by N_{c2} D4 branes. As discussed in [10], the sign of the $A^{(5)}$ charge of the orientifold (which is related to the sign of Ω^2) flips as one passes a NS5 brane and flips back again as one passes other NS5 brane. If the sign of $A^{(5)}$ is chosen to be negative between A 5 brane and B 5 brane, it will be positive between B 5 brane and C 5 brane. For this reason the product gauge group becomes $SO(N_{c1}) \times Sp(N_{c2})$ (or $Sp(N_{c1}) \times SO(N_{c2})$) if we invert the overall sign of the projection). We will use these observations in our procedure. In other words, the choice of the gauge group SO or Sp results from the sign of Ω^2 on the open string sectors. Between A(B) 5 brane and B(C) 5 brane we have $2N_{f1}(2N_{f2})$ D6 branes intersecting the $N_{c1}(N_{c2})$ D4 branes. Strings stretching between the $2N_{f1}(2N_{f2})$ D6-branes and the $N_{c1}(N_{c2})$ D4 branes are the chiral multiplets in the vector(fundamental) representation of $SO(N_{c1})(Sp(N_{c2}))$. Strings can also stretch between N_{c1} D4 branes and N_{c2} D4 branes. We orient the $2N_{f1}(2N_{f2})$ D6 branes in the direction parallel to the A(C) 5 brane so there exist chiral multiplets which correspond to the motion of D4 branes in between the NS5 and D6 branes, as discussed in [11]. These states are precisely the chiral mesons of the magnetic dual theory which will be obtained later.

For $SO(N_{c1})$ group, we will consider only $N_{c1}/2$ D4 branes, the other half being their mirrors. For $Sp(N_{c2})$ group, because of the antisymmetric O4 projections, we have to consider a total even number of D4 branes, i.e., N_{c2} D4 branes and their orientifold mirrors. When strings are stretched between the N_{c1} and N_{c2} D4 branes, a field X in the $(N_{c1}, 2N_{c2})$ representation of product gauge group $SO(N_{c1}) \times Sp(N_{c2})$ is obtained and the superpotential $W = \text{Tr}X^4$ appears in order to truncate the chiral ring. A nice way to view its appearance is to look at our theory from the point of view of the original $N = 2$ supersymmetric theory. As discussed in [11, 15], when all NS5 branes are parallel we have an $N = 2$ SUSY theory while as we rotate the NS5 branes, $N = 2$ theory is broken and part of the $N = 2$ superpotential appears as a superpotential in $N = 1$ supersymmetric theory. Therefore the electric theory is just the one given in field theory approach [46].

[§]Note that we denote $2N_{f1}$ and $2N_{f2}$ by flavors in each representation rather than N_{f1} and N_{f2} which were used in the paper of [46] in order to avoid the fractional expression when we consider its mirrors. We use for simplicity that each factor appearing in the flavors, representations and gauge groups, corresponds to its own order.

The original(electric) picture is the following from left to right: Between A 5 brane and B 5 brane we have $N_{c1}/2$ D4 branes intersecting N_{f1} D6 branes and between B 5 brane and C 5 brane we have N_{c2} D4 branes intersecting N_{f2} D6 branes(plus their mirrors).

In order to find the dual theory, like in [11], first move all the N_{f1} D6 branes to the right of all NS5 branes. They are intersecting both B 5 brane with C 5 brane. Using the “linking number” conservation argument (the linking number for a particular brane must be conserved), it turns out that each D6 brane has two D4 branes on its left after transition. Similarly by pushing all N_{f2} D6 branes to the left, they are crossing both B 5 brane and A 5 brane and each D6 brane has two D4 branes on its right after transition.

Because the NS5 branes are trapped at the spacetime orbifold fixed point, they cannot avoid their intersection so they should meet and there exists a strong coupling singularity when they are intersecting. When each of B 5 brane and C 5 brane actually meets A 5 brane, such a singularity appears. From the field theory point of view, such a non smooth behavior was expected because for $Sp(N_c)$ and $SO(N_c)$ gauge groups there is a phase transition unlike for $SU(N_c)$ gauge group. In [10], the effect of such a singularity was proved to be the appearance or disappearance of two D4 branes. In [13] the transition was shown to be smooth only when the linking number of both sides of any NS5 brane is the same. For SO gauge group, the procedure is to *put* two of the N_{c1} D4 branes on top of the orientifold plane and *break* the other $(N_{c1} - 2)$ D4 branes on D6 branes , entering in a Higgs phase. There is no problem to vary smoothly the separation between the two NS5 branes in x^6 direction. On the other hand, for Sp gauge group a pair of D4 branes and anti D4 brane plus their mirrors are created and anti D4 branes cancel the charge difference along the orientifold.

Let us go on to the magnetic dual theory. First we move C 5 brane to the left of B 5 brane. In the language of [10], two D4 branes must disappear because we have Sp gauge group between C 5 brane and B 5 brane. Then we move C 5 brane to the left of A 5 brane. When C 5 brane passes A 5 brane, two D4 branes appear between A 5 brane and C 5 brane because we have an SO gauge group, so totally we have a deficit of two D4 branes between A 5 brane and B 5 brane and no extra D4 branes between C 5 brane and A 5 brane. Now push B 5 brane to the right of A 5 brane. When B 5 brane passes A 5 brane (because of the Sp group) there is an extra deficit of two D4 branes, but the initial supplementary D4 branes are changing the orientation and so we have no extra D4 branes in the theory. In the language of [13], for a smooth transition between B 5 brane and C 5 brane, we need to create two pairs of D4 branes and anti D4 branes, the anti D4 branes neutralizing the charge difference along the orientifold as before. The two D4 branes to be put on top of the orientifold, when C 5 brane passes A 5 brane and B 5 brane passes A 5 brane, annihilate the anti D4 branes. The two D4 branes which were on top of the orientifold, came from the N_{c1} D4 branes connecting A

5 brane and B 5 brane therefore leaving $(N_{c1} - 2)$ D4 branes between A 5 brane and B 5 brane. Therefore the two D4 branes, which remain after their anti D4 branes vanish, are added to the $(N_{c1} - 2)$ D4 branes. So by smoothing the transition, we did not produce any D4 or anti D4 branes, as expected from the field theory calculation.

The final(magnetic) picture is the following, from right to left: N_{f1} D6 branes are connected by $2N_{f1}$ D4 branes with A 5 brane. Between A 5 brane and B 5 brane we have \widetilde{N}_{c2} D4 branes, between B 5 brane and C 5 brane we have $\widetilde{N}_{c1}/2$ D4 branes and to the left of C 5 brane we have N_{f2} D6 branes connected by $2N_{f2}$ branes with C 5 brane(plus their mirrors).

We use the linking number of A 5 brane to calculate the magnetic Sp gauge group, \widetilde{N}_{c2} . In the original electric picture it was equal to $-N_{f1}/2 - N_{f2}/2 + N_{c1}/2$ where we followed conventions: NS5 branes or D6 branes to the left(right) of the brane we are considering give a contribution of $1/2(-1/2)$ to the linking number while D4 branes to the left(right) give a contribution of $-1(1)$. In the magnetic picture the linking number becomes $-N_{f1}/2 + N_{f2}/2 - \widetilde{N}_{c2} + 2N_{f1}$. By making them equal, we obtain $\widetilde{N}_{c2} = 2N_{f1} + N_{f2} - N_{c1}/2$. For the B 5 brane, the conservation of the linking number gives the relation $-N_{f2}/2 + N_{f1}/2 + N_{c2} - N_{c1}/2 = \widetilde{N}_{c2} - \widetilde{N}_{c1}/2 - N_{f1}/2 + N_{f2}/2$. Then we obtain $\widetilde{N}_{c1} = 2N_{f1} + 4N_{f2} - 2N_{c2}$. The values for \widetilde{N}_{c1} and \widetilde{N}_{c2} coincide with precisely the ones obtained in [46] by remembering that we started with even number of flavors $2N_{f1}$ and $2N_{f2}$. The magnetic theory obtained by inverting the order of NS5 branes can be described as a theory with the gauge group $SO(\widetilde{N}_{c1}) \times Sp(\widetilde{N}_{c2})$ with $\widetilde{N}_{c1} = 2N_{f1} + 4N_{f2} - 2N_{c2}$ and $\widetilde{N}_{c2} = 2N_{f1} + N_{f2} - N_{c1}/2$. From the brane configuration discussed above, the field contents of the theory are as follows: the gauge group $SO(\widetilde{N}_{c1}) \times Sp(\widetilde{N}_{c2})$ with $2N_{f2}(2N_{f1})$ fields in the vector(fundamental) representation of $SO(\widetilde{N}_{c1})(Sp(\widetilde{N}_{c2}))$ and a field Y in the $(\widetilde{N}_{c1}, 2\widetilde{N}_{c2})$ representation of the product gauge group where the chiral mesons of the dual theory which have the same form as in [46].

Now we want to describe the above duality in the context of OV approach and to see what is the correspondence between them. In [21], they obtained an $N = 1$ theory in $d = 4$ living on the (3+1)-part of the worldvolume of D6 branes by partially wrapping the D6 branes around vanishing real 3-cycles in a local model of a doubly elliptic fibered Calabi-Yau threefold in type IIA string theory.

In order to study $SO(N_{c1}) \times Sp(N_{c2})$ gauge theories with flavors, we consider a twisted local model of a doubly elliptic fibered CY threefold given by

$$\begin{aligned} (x &+ [m_1 + m_2(\mu_1 z - \mu_2)(z - a_1)(z - a_2)]x')^2 + \\ (y &+ [m_1 + m_2(\mu_1 z - \mu_2)(z - a_1)(z - a_2)]y')^2 = -(z - a_1)(z - a_2)(z - c_2)(z - c_1) \\ x'^2 &+ y'^2 = -z \end{aligned}$$

$$\begin{aligned}\mu_1 &= \frac{1}{c_1} \left\{ \frac{1}{(c_1 - a_1)(c_1 - a_2)} - \frac{1}{2} \left[\frac{1}{(c_1 - a_1)(c_1 - a_2)} - \frac{1}{(c_2 - a_1)(c_2 - a_2)} \right] \right\} \\ \mu_2 &= \frac{1}{2} \left[\frac{1}{(c_1 - a_1)(c_1 - a_2)} - \frac{1}{(c_2 - a_1)(c_2 - a_2)} \right]\end{aligned}\quad (1)$$

where a_i, c_i 's are real numbers with $a_1 < a_2 < 0 < c_2 < c_1$. Here m_1 and m_2 are arbitrary real numbers. When they are zero, we get back the usual local model of a CY threefold. Via the projection $(x, y, x', y', z) \rightarrow z$, we may regard the above local description as an approximation of a doubly elliptic fibered CY threefold over the z -plane near its degeneration. The general fiber is isomorphic to $\mathbf{C}^* \times \mathbf{C}^*$, but we cannot split the fibers uniformly (i.e. independent of z) into a product of two \mathbf{C}^* because the first \mathbf{C}^* is not embedded in (x, y) -space even though the second \mathbf{C}^* is still embedded in (x', y') -space. The CY threefold is smooth, but the fiber over the z -plane acquires A_1 singularities as z approaches a_i, c_i or 0. For a fixed z away from a_i, c_i or 0, there exist nontrivial S^1 's in each of \mathbf{C}^* which vanishes as z approaches to a_i, c_i or 0 so that the union of these vanishing S^1 's over a real line segment near a_i, c_i or 0 will look like a 'thimble'. Thus the cycles over the real line segments $[a_2, a_1], [c_2, c_1]$ are S^3 's and the cycles over the real line segment $[a_1, 0], [0, c_2]$ are $S^2 \times S^1$'s.

Now we orientifold this configuration by combining the complex conjugation

$$(x, y, x', y', z) \rightarrow (x^*, y^*, x'^*, y'^*, z^*) \quad (2)$$

with exchange of left- and right-movers on the worldsheet. Thus the orientifold 6-space (i.e. the fixed point set) is the real CY threefold defined by the equation (1) times the uncompactified (x_0, x_1, x_2, x_3) space. As it is shown in [21], one can see that the degeneration of the fiber at each a_i, c_i or 0 is replaced by one NS 5 brane after performing the T-duality on each \mathbf{C}^* of the double elliptic fibration. We choose coordinates so that NS 5 branes arising at a_i or c_i 's when $m_i = 0$ in the equation (1) are parallel to the $x^0, \dots, x^3, x^4, x^5$ plane, and NS 5 brane arising at 0 is parallel to the $x^0, \dots, x^3, x^8, x^9$ plane. But for generic m_i , NS 5 branes at a_i 's are parallel but not parallel to NS 5 brane at 0 or c_i 's. Also NS 5 branes at c_i 's are parallel, but not parallel to NS 5 brane at 0 or a_i 's.. Being common transverse direction to all NS 5 branes, (x^6, x^7) will be regarded as real and imaginary parts of z coordinate.

The contribution of the O6 plane to the D6 brane charge is determined by noting that O9 plane carries (-16) units of D9 charge (See, for example, [47]). Then we compactify on a three torus T^3 and T-dualize. We obtain eight O6 planes with the same total charge as before, that is, (-16) units of D6 brane charge. Therefore, each O6 plane carries (-2) units of D6 brane charge. Because the contribution of the O6 plane to D6 brane charge comes from a diagram with a single crosscap, going from SO to Sp involves the sign change for diagrams with odd number of crosscaps. This means O6 plane will carry $+2$ units of D6 brane charge for the Sp case.

We consider $(N_{c_1}/2 - 2)$ D6 brane charge on the cycle $[a_1, a_2]$, $(N_{c_2} + 2)$ D6 brane charge on the cycle $[c_2, c_1]$, N_{f_1} D6 brane charge on the cycle $[a_2, 0]$ and N_{f_2} D6 brane charge on the cycle $[0, c_1]$ as in the following Figure 1.

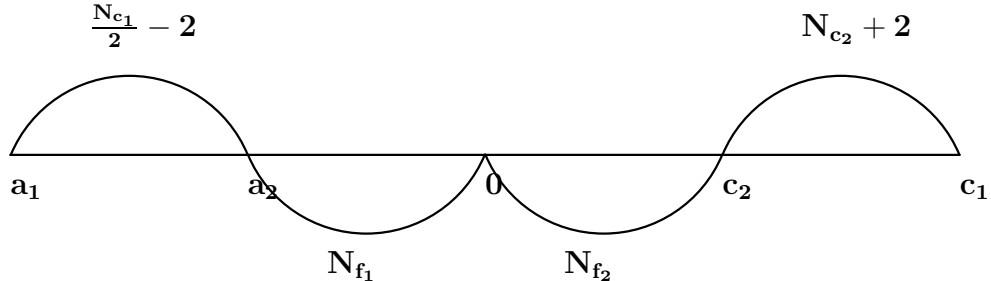


Figure 1:

To make the flavor groups global we need to push a_2 and c_2 to infinity. To do that we have to push a_2 to the right (resp. c_2 to the left) since NS branes at a_i 's (resp. c_i 's) are parallel in the T-dual picture. The main geometric difficulty here is that we cannot lift the degeneration points off the real axis because it is frozen by the orientifolding. As two of these degeneration points collide, it will create a singular CY threefold. Physically, this means that we have to go through a strong coupling region. Rather than directly investigating the situation, we perform T duality and appeal to the linking number argument as before.

We now have five NS5 branes placed at each of the previous five points. The ones at a_1 and a_2 (resp. c_1 and c_2) are parallel and are not parallel to any others. The NS5 branes at a_2 and c_2 play the role of the N_{f_1} and N_{f_2} D6 branes respectively in the brane configuration picture we have seen where by moving N_{f_1} D6 brane from right to left with respect to a NS5 brane, N_{f_1} D4 brane will appear to the left of the N_{f_1} D6 brane. Our claim here is that when we move the NS5 branes sitting at c_2 from right to left with respect to the one sitting at 0 or a_2 , the same amount of D4 branes will appear. But no more D4 branes will be created when c_2 passes thru from a_2 to a_1 because the cycle type has changed once c_2 passes through a_2 and the NS branes at a_1 and a_2 are parallel. The creation of D4 branes has been explained in [21] geometrically. We will give another possible explanation in the appendix.

Going back (by T-duality) to the original picture, this means that when we move c_2 to the left of a_i and 0 then $2N_{f_2}$ supplementary D6 branes are wrapped on $[c_2, a_1]$. The same argument tells us that when we move a_2 to the right of c_1 , then $2N_{f_1}$ supplementary D6 branes are wrapped on $[c_1, a_2]$. Now we want to move to other point in the moduli of the CY threefolds and end up with a configuration where again the degeneration points are along the real axis in the z -plane, but the order is changed to $(c_2, c_1, 0, a_1, a_2)$. We decide the number of D6 branes wrapping on $[c_1, 0]$ and $[0, c_2]$ by linking number argument.

Thus we obtain the final configuration of points ordered as $(c_2, c_1, 0, a_1, a_2)$ with $2N_{f2}$ D6 branes wrapped on $[c_2, c_1]$, $(N_{f1} + 2N_{f2} - N_{c2} - 2)$ D6 branes wrapped on $[c_1, 0]$, $(2N_{f1} + N_{f2} - N_{c1}/2 + 2)$ D6 branes wrapped on $[0, a_1]$ and $2N_{f1}$ D6 branes wrapped on $[a_1, a_2]$ as in the Figure 2.

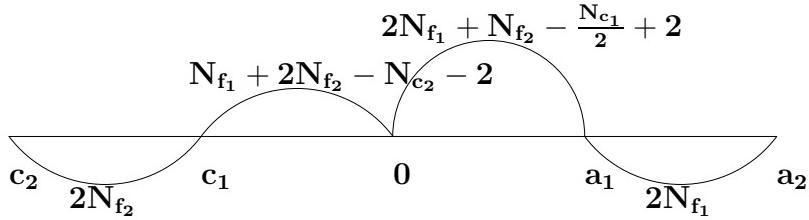


Figure 2:

This picture describes the magnetic dual theory which has a gauge group $SO(2N_{f1} + 4N_{f2} - 2N_{c2}) \times Sp(2N_{f1} + N_{f2} - N_{c1}/2)$ with $2N_{f2}(2N_{f1})$ flavors in the vector(fundamental) representation of the $SO(Sp)$ gauge group. In order to obtain global flavor groups, we push a_2 and c_2 to infinity. We have singlets which correspond to the mesons of the original electric theory and interact with the dual quarks through the superpotential in the magnetic theory.

2.2 The superpotential $\mathbf{W} = \text{Tr } X^{4(k+1)}$ case

We now move to a more general form of superpotential. In brane configuration, it is straightforward to show that the previous arguments can be generalized by taking multiple copies of NS5 branes.

This theory corresponds to (from left to right) $(2k+1)$ NS5 branes with the orientation of C connected to a single B 5 brane by N_{c2} D4 branes. The single B 5 brane is connected to $(2k+1)$ NS5 branes with the same orientation as A by $N_{c1}/2$ D4 branes. N_{f1} and N_{f2} D6 branes intersect the N_{c1} and N_{c2} D4 branes, respectively (plus their mirrors).

As done in previous subsection, we move all the N_{f1} D6 branes to the left of all NS5 branes. They intersect both B 5 brane and $(2k+1)$ C 5 branes. Using the linking number conservation argument each D6 brane has $2(k+1)$ D4 branes on its right after the transition. Similarly, by moving all N_{f2} D6 branes to the right, they are intersecting both B 5 brane and $(2k+1)$ A 5 brane and from the conservation of linking number each D6 brane has $2(k+1)$ D4 branes on its left after transition.

The final picture is the following, from left to right: N_{f1} D6 branes connected by

$2(k+1)N_{f1}$ D4 branes with $(2k+1)$ A 5 branes. Between $(2k+1)$ A 5 branes and B 5 brane we have \widetilde{N}_{c2} D4 branes, between B 5 brane and $(2k+1)$ C 5 branes we have $\widetilde{N}_{c1}/2$ D4 branes and to the right of $(2k+1)$ C 5 branes we have N_{f2} D6 branes, connected by $2(k+1)N_{f2}$ D4 branes with $(2k+1)$ C 5 branes (plus their mirrors).

We apply the linking number conservation argument for A 5 brane to calculate \widetilde{N}_{c2} . In the original electric picture it is equal to $-(2k+1)N_{f1}/2 - (2k+1)N_{f2}/2 + N_{c1}/2$. In the magnetic picture the linking number is $-(2k+1)N_{f1}/2 + (2k+1)N_{f2}/2 - \widetilde{N}_{c2} + 2(k+1)N_{f1}$. Making them equal, we obtain $\widetilde{N}_{c2} = 2(k+1)N_{f1} + (2k+1)N_{f2} - N_{c1}/2$. For the B 5 brane, the conservation of the linking number gives $-N_{f2}/2 + N_{f1}/2 + N_{c2} - N_{c1}/2 = \widetilde{N}_{c2} - \widetilde{N}_{c1}/2 - N_{f1}/2 + N_{f2}/2$. We obtain $\widetilde{N}_{c1} = 2(2k+1)N_{f1} + 4(k+1)N_{f2} - 2N_{c2}$. The values for \widetilde{N}_{c1} and \widetilde{N}_{c2} coincide with the ones discussed in [46] exactly.

In the geometrical picture, after a T-duality, this would mean that instead of having single NS 5-brane at a_1 we have $(2k+1)$ NS5 branes and instead of having one NS 5-brane at c_1 we have $(2k+1)$ NS5 branes. In order to handle this problem we may regard $(2k+1)$ NS5 branes at one point as $(2k+1)$ different points where each NS5 brane corresponds to each point. This point of view was introduced in [24] for the gauge group $SU(N_c)$ with adjoint matter. For simplicity, let us consider first the case $k=1$ (Remind that the theory of the previous subsection corresponds to the case of $k=0$). We have to twist the doubly elliptic fibration again so that in T-dual picture only the branes a_{11}, a_2 (resp. c_2, c_{11}) are parallel and all others are at arbitrary angles.

On the real axis we will have from left to right: $(a_{13}, a_{12}, a_{11}, a_2, 0, c_2, c_{11}, c_{12}, c_{13})$, where instead of a_1 (resp. c_1) we have a_{11}, a_{12}, a_{13} (resp. c_{11}, c_{12}, c_{13}). Using the idea of [24, 25], we have D6 brane charges $(N_{c1}/2 - 2)$ on $[a_{11}, a_2]$, $(N_{c2} + 2)$ on $[c_2, c_{11}]$. Besides we also have N_{f1} D6 brane charge on $[a_2, 0]$ and N_{f2} D6 brane charge on $[0, c_2]$ and a single D6 brane charge on $[a_{13}, a_{12}], [a_{12}, a_{11}], [c_{11}, c_{12}]$ and $[c_{12}, c_{13}]$ as in Figure 3.

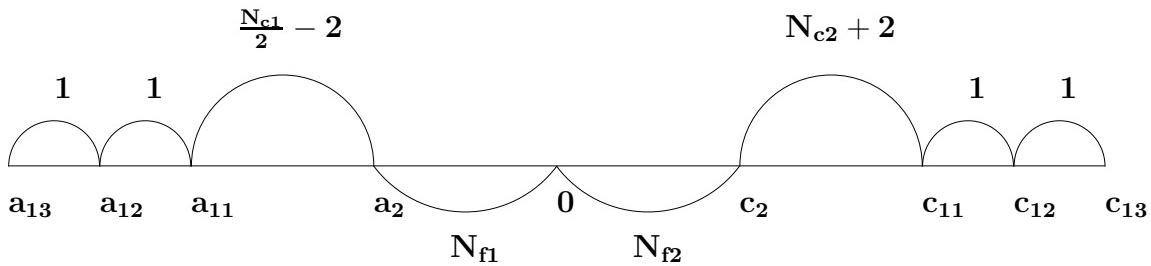


Figure 3:

Now we move c_2 to the left of a_{11}, a_{12}, a_{13} and a_2 to the right of c_{11}, c_{12}, c_{13} . To apply the observation made in the previous subsection, we perform first a T-duality in (x^4, x^5, x^8, x^9) directions. In the T-dual language, moving c_2 to the left of a_{11}, a_{12}, a_{13}

means moving the NS5 brane sitting at c_2 to the left with respect to the NS5 branes sitting at a_{11}, a_{12} and a_{13} . When c_2 passes $0, a_2$ and $a_{1i}(i = 2, 3)$, N_{f2} supplementary D6 branes are created. Going back to the original geometrical picture, we have $4N_{f2}$ D6 branes wrapped on $[c_2, a_{13}]$. The same thing appears when we move a_2 to the right of c_{11}, c_{12} and c_{13} . Note that the branes between a_{1i} 's or c_{1i} 's disappear in the limit $a_{11} = a_{12} = a_{13}$ or $c_{11} = c_{12} = c_{13}$. We now want to move to another point of moduli space of the CY threefold by changing the positions of $a_{11} = a_{12} = a_{13} = a_1$ and $c_{11} = c_{12} = c_{13} = c_1$. We push c_1 and 0 to the right of a_1 to obtain the magnetic theory. The numbers of D6 branes wrapped on $[c_1, 0]$ and $[0, a_1]$ are decided by their linking numbers.

The final configuration is $(c_2, c_1, 0, a_1, a_2)$ with $4N_{f2}$ D6 branes wrapped on $[c_2, c_1]$, $(3N_{f1} + 4N_{f2} - N_{c2} - 2)$ D6 branes wrapped on $[c_1, 0]$, $(4N_{f1} + 3N_{f2} - N_{c1}/2 + 2)$ D6 branes wrapped on $[0, a_1]$ and $4N_{f1}$ D6 branes wrapped on $[a_1, a_2]$ as in Figure 4.

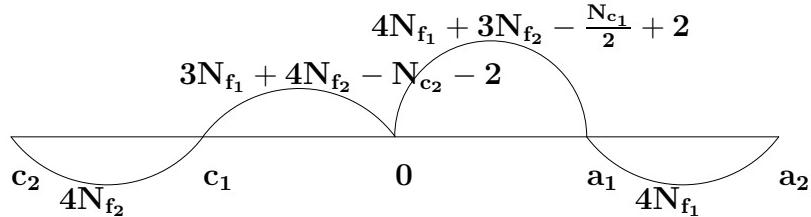


Figure 4:

This gives a field theory with the gauge group $SO(2(3N_{f1} + 4N_{f2}) - 2N_{c2}) \times Sp(4N_{f1} + 3N_{f2} - N_{c1}/2)$ with $2N_{f2}(2N_{f1})$ flavors in the vector(fundamental) of the $SO(\widetilde{N}_{c1})(Sp(\widetilde{N}_{c2}))$ gauge group where \widetilde{N}_{c1} and \widetilde{N}_{c2} are the same as those for $k = 1$ given in brane configuration.

The generalization to arbitrary k becomes now obvious. Instead of $a_1(c_1)$ we take $(2k+1)$ singular points like as $a_{11}, a_{12}, \dots, a_{1(2k+1)}(c_{11}, c_{12}, \dots, c_{1(2k+1)})$. The points $a_2, 0$ and c_2 will still remain in a single copy. Now we wrap a single D6 brane on $[a_{1i}, a_{1i+1}]$ for each i and on $[c_{1j}, c_{1j+1}]$ for each j . Also we wrap N_{f1} on $[0, a_2]$ and N_{f2} on $[c_2, 0]$. When we move $a_{11}, \dots, a_{1(2k+1)}$ to the right of c_2 , for every transition, N_{f2} supplementary D6 branes will appear. Eventually we have $2(k+1)N_{f2}$ D6-branes wrapped on $[c_2, a_{11}]$ and so on. The same thing happens when we push $c_{11}, \dots, c_{1(2k+1)}$ to the left of a_2 and we end up with $2(k+1)N_{f1}$ branes wrapped on $[c_{11}, a_2]$. Now we identify $a_{1i}(c_{1j})$ with $a_1(c_1)$ like as $a_{11} = \dots = a_{1(2k+1)} = a_1$ and $c_{11} = \dots = c_{1(2k+1)} = c_1$ and change again the positions of 0 and c_1 from the right to the left of a_1 to obtain the magnetic dual description. This has a gauge group $SO(2(2k+1)N_{f1} + 4(k+1)N_{f2} - 2N_{c2}) \times Sp(2(k+1)N_{f1} + (2k+1)N_{f2} - N_{c1}/2)$ which is exactly the same as that obtained in brane configuration.

2.3 The addition of adjoint matter

Now we can go on one further step by taking multiple copies of the middle NS5 brane. In brane configuration, we claim that the model with adjoint matter is described by $(2k+1)$ A 5 branes connected to $(2k+1)$ B 5 branes by N_{c1} D4 branes. The $(2k+1)$ B 5 branes are connected to $(2k+1)$ C 5 branes by N_{c2} D4 branes. The N_{f1} and N_{f2} D6 branes intersect the N_{c1} and N_{c2} D4 branes respectively.

We use the linking number of A 5 brane to calculate \widetilde{N}_{c2} as usual. In the original electric picture it was equal to $-(2k+1)N_{f1}/2 - (2k+1)N_{f2}/2 + N_{c1}/2$. In the magnetic picture the linking number can be written as $-(2k+1)N_{f1}/2 + (2k+1)N_{f2}/2 - \widetilde{N}_{c2} + 2(2k+1)N_{f1}$. The conservation of linking number allows us to have $\widetilde{N}_{c2} = 2(2k+1)N_{f1} + (2k+1)N_{f2} - N_{c1}/2$. For the B 5 brane, the conservation of the linking number gives the relation $-(2k+1)N_{f2}/2 + (2k+1)N_{f1}/2 + N_{c2} - N_{c1}/2 = \widetilde{N}_{c2} - \widetilde{N}_{c1}/2 - (2k+1)N_{f1}/2 + (2k+1)N_{f2}/2$. From this we obtain $\widetilde{N}_{c1} = 2(2k+1)N_{f1} + 4(2k+1)N_{f2} - 2N_{c2}$.

We now go to the geometrical picture and see how the geometric configuration appears. Let us take $k=1$ for simplest case: $(a_{13}, a_{12}, a_{11}, a_2, b_1, 0, b_2, c_2, c_{11}, c_{12}, c_{13})$. Note that a_2 and c_2 are always in a single copy. We now wrap $(N_{c1}/2 - 2)$ D6 brane charge on $[a_{11}, a_2]$, N_{f1} on $[a_2, b_1]$, 1 on $[b_1, 0]$, 1 on $[0, b_2]$, N_{f2} on $[b_2, c_2]$, $(N_{c2} + 2)$ on $[c_2, c_{11}]$, 1 on $[c_{11}, c_{12}]$ and 1 on $[c_{12}, c_{13}]$. If we take the limit $b_1, b_2 \rightarrow 0$, we have now $3N_{f1}$ D6 branes wrapped on $[a_2, b_1 = b_2 = 0]$ and $3N_{f2}$ D6 branes on $[b_1 = b_2 = 0, c_2]$. Again the configuration looks very much alike the previous configurations. We draw the schematic picture in the Figure 5.

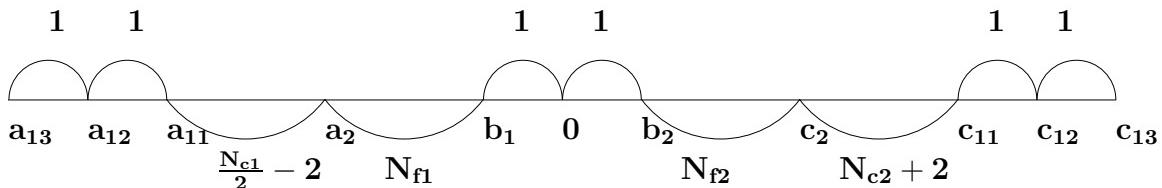


Figure 5:

We now move c_2 to the left of a_{11}, a_{12} and a_{13} and a_2 to the right of c_{11}, c_{12} and c_{13} . In this case, we apply again the observation about the creation of supplementary D6 branes we have discussed before. We obtain $6N_{f2}$ D6 branes wrapped on $[c_2, a_{13}]$ and $6N_{f1}$ wrapped on $[c_{13}, a_2]$.

We now move to other point in the moduli of CY threefolds and end up with the magnetic dual theory. The procedure is the same as before and we obtain the magnetic theory with the gauge group $SO(6N_{f1} + 12N_{f2} - 2N_{c2}) \times Sp(6N_{f1} + 3N_{f2} - N_{c1}/2)$ as in Figure 6.

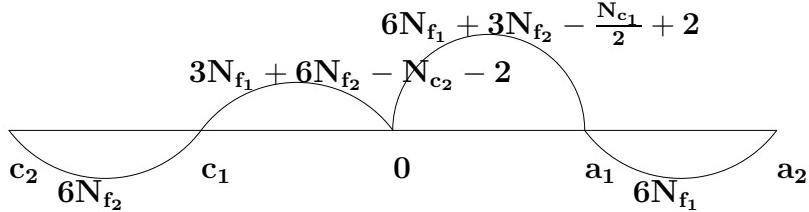


Figure 6:

In order to generalize to the case of arbitrary k , we take $(2k+1)$ copies of all the singular points $a_1, 0$ and c_1 . When all these copies coincide, then we will have $(2k+1)N_{f2}$ D6 branes wrapped on $[c_2, a_1]$ and $(2k+1)N_{f1}$ wrapped on $[c_1, a_2]$. The electric theory has the following configuration of singular points $(a_2, a_{11} = \dots = a_{1(2k+1)} = a_1, b_1 = \dots = b_k = 0 = b_{k+1} = \dots = b_{2k+1}, c_{11} = \dots = c_{1(2k+1)} = c_1, c_2)$, with $2(2k+1)N_{f2}$ D6 branes wrapped on $[a_2, a_1]$, $((2k+1)N_{f2} + N_{c1}/2 - 2)$ D6 branes wrapped on $[a_1, 0]$, $((2k+1)N_{f1} + N_{c2} + 2)$ D6 branes wrapped on $[0, c_1]$ and $2(2k+1)N_{f1}$ D6 branes wrapped on $[c_1, c_2]$. The magnetic dual theory can be obtained by changing the position of 0 and c_1 from right to left with respect to a_1 . Finally we get the magnetic theory with the gauge group $SO(2(2k+1)N_{f1} + 4(2k+1)N_{f2} - 2N_{c2}) \times Sp(2(2k+1)N_{f1} + (2k+1)N_{f2} - N_{c1}/2)$ which again agrees with the one obtained in brane configuration for any value of k .

Let us discuss the duality for adjoint matter in the field theory setup. Besides the field X that we had in the previous subsections we introduce adjoint fields for both gauge groups of the product gauge group. We denote by $X_1(X_2)$ the antisymmetric(symmetric) adjoint fields of $SO(N_{c1})(Sp(N_{c2}))$. The matter content of the electric theory is given by (we will consider only the gauge groups and $U(1)_R$ charges for the matter fields):

	$SO(N_{c1})$	$Sp(N_{c2})$	$U(1)_R$
$Q^{(1)}$	\mathbf{N}_{c1}	$\mathbf{1}$	$1 + \frac{1}{N_{f1}}(\frac{N_{c2}-N_{c1}}{k+1} + 2)$
$Q^{(2)}$	$\mathbf{1}$	$2\mathbf{N}_{c2}$	$1 + \frac{1}{N_{f2}}(\frac{N_{c1}-4N_{c2}}{2(k+1)} - 2)$
A_1	$(\mathbf{N}_{c1} - \mathbf{1})\mathbf{N}_{c1}/2$	$\mathbf{1}$	$\frac{k+2}{2(k+1)}$
A_2	$\mathbf{1}$	$2\mathbf{N}_{c2}(2\mathbf{N}_{c2} + \mathbf{1})/2$	$\frac{k+2}{2(k+1)}$
X	\mathbf{N}_{c1}	$2\mathbf{N}_{c2}$	$\frac{k}{2(k+1)}$

The superpotential for the theory is

$$W = \text{Tr}A_1^{2(k+1)} + \text{Tr}A_2^{2(k+1)} + \text{Tr}A_1X^2 - \text{Tr}A_2X^2.$$

The chiral ring truncates because of this superpotential, in the sense that the maximum power of A_i that can appear is k . The gauge invariant mesons in the theory are

$$Q^{(1)} A_1^j Q^{(1)}, Q^{(2)} A_2^j Q^{(2)}, Q^{(1)} A_1^j X Q^{(2)}, Q^{(1)} A_2^j X Q^{(2)}, Q^{(1)} X^2 A_1^j Q^{(1)}, Q^{(2)} X^2 A_2^j Q^{(2)}$$

where $j = 1, \dots, k - 1$.

The dual theory is described by an $SO(2(2k + 1)N_{f1} + 4(2k + 1)N_{f2} - 2N_{c2}) \times Sp(2(2k + 1)N_{f1} + (2k + 1)N_{f2} - N_{c1}/2)$ gauge theory with the following charged matter contents:

	$SO(\widetilde{N}_{c1})$	$Sp(\widetilde{N}_{c2})$	$U(1)_R$
$q^{(1)}$	$\widetilde{\mathbf{N}}_{\mathbf{c1}}$	$\mathbf{1}$	$1 + \frac{1}{N_{f1}}(\frac{\widetilde{N}_{c2} - \widetilde{N}_{c1}}{k+1} + 2)$
$q^{(2)}$	$\mathbf{1}$	$2\widetilde{\mathbf{N}}_{\mathbf{c2}}$	$1 + \frac{1}{N_{f2}}(\frac{\widetilde{N}_{c1} - 4\widetilde{N}_{c2}}{2(k+1)} - 2)$
\tilde{A}_1	$(\widetilde{\mathbf{N}}_{\mathbf{c1}} - \mathbf{1})\widetilde{\mathbf{N}}_{\mathbf{c1}}/\mathbf{2}$	$\mathbf{1}$	$\frac{k+2}{2(k+1)}$
\tilde{A}_2	$\mathbf{1}$	$2\widetilde{\mathbf{N}}_{\mathbf{c2}}(2\widetilde{\mathbf{N}}_{\mathbf{c2}} + \mathbf{1})/\mathbf{2}$	$\frac{k+2}{2(k+1)}$
Y	$\widetilde{\mathbf{N}}_{\mathbf{c1}}$	$2\widetilde{\mathbf{N}}_{\mathbf{c2}}$	$\frac{k}{2(k+1)}$

where $\widetilde{N}_{c1} = 2(2k + 1)N_{f1} + 4(2k + 1)N_{f2} - 2N_{c2}$ and $\widetilde{N}_{c2} = 2(2k + 1)N_{f1} + (2k + 1)N_{f2} - N_{c1}/2$. In the dual theory, the images of the mesons coming from the electric field theory are singlet fields and the dual superpotential is obtained by adding coupling terms between singlets and dual mesons to the superpotential of the electric theory.

3 $SO(N_{c1}) \times Sp(N_{c2}) \times SO(N_{c3})$

Since we have seen a brane configuration containing two sets of D4 branes suspended between three NS5 branes, it is natural to ask what happens for a brane configuration including more D4 branes suspended between more than three NS5 branes. We expect to see *new* dualities in the context of brane configuration which have not been studied before even at the field theory level.

3.1 The theory with simplest superpotential

Let us study triple product gauge group for the electric theory for the simplest case where each of the four kinds of NS5 branes has its single copy. The gauge group is $SO(N_{c1}) \times$

$Sp(N_{c2}) \times SO(N_{c3})$ with $2N_{f1}(2N_{f2})[2N_{f3}]$ flavors in the vector (fundamental)[vector] representation of the respective $SO(N_{c1})(Sp(N_{c2}))[SO(N_{c3})]$ gauge group. We label them by letters A, B, C and D from left to right on the compact x^6 direction. The four NS5 branes are oriented at arbitrary angles in (x^4, x^5, x^8, x^9) directions. None of these NS5 branes are parallel.

There exist $N_{c1}/2$ D4 branes stretched between A 5 brane and B 5 brane which is connected to C 5 brane by N_{c2} D4 branes. The C 5 brane is connected to D 5 brane by $N_{c3}/2$ D4 branes. Between A(B)[C] 5 brane and B(C)[D] 5 brane we have $N_{f1}(N_{f2})[N_{f3}]$ D6 branes intersecting the $N_{c1}/2(N_{c2})[N_{c3}/2]$ D4 branes(plus their mirrors).

In order to find magnetic dual theory, it is easy to see from the similar previous arguments that by moving all the N_{f1} D6 branes to the left of all NS5 branes, each D6 brane has three D4 branes on its left after transition and by moving all N_{f3} D6 branes to the right, each D6 brane has three D4 branes on its left after transition. First move B 5 brane to the right of A 5 brane. Two D4 branes must appear because we have SO gauge group according to [10] or two D4 branes must be stuck on the orientifold O4 plane in the language of [13]. We move C 5 brane to the right of A 5 brane. When C 5 brane passes A 5 brane, two D4 branes disappear between A 5 brane and C 5 brane because we have an Sp gauge group, so we have two D4 branes between C 5 brane and B 5 brane and no extra D4 branes between A 5 brane and C 5 brane. Now move D 5 brane to the right of A 5 brane. When D 5 brane passes A 5 brane there appear two new D4 branes between A 5 brane and D 5 brane. In this moment we have two supplementary D4 branes between A 5 brane and D 5 brane, no supplementary D4 branes between D 5 brane and C 5 brane and two supplementary D4 branes between C 5 brane and B 5 brane. Similarly we move C 5 brane to the right of B 5 brane there are two new D4 branes because of the SO gauge group, but the previous supplementary two D4 branes will change their orientations, they will cancel the new two D4 branes which leads eventually to no supplementary or missing D4 branes between B 5 brane and C 5 brane. Now move D 5 brane to the right of B 5 brane, then we have minus two D4 branes. After moving D 5 brane to the right of C 5 brane we finally get plus(minus)[plus] two D4 branes between A(B)[C] 5 brane and B(C)[D] 5 brane.

The final configuration is the following, from left to right: semiinfinite N_{f2} D4 branes ending on A 5 brane and semiinfinite N_{f2} D4 branes ending on B 5 brane. There are $3N_{f1}$ between N_{f1} D6 branes and A 5 brane which is connected to B 5 brane. There are $\widetilde{N}_{c3}/2$ D4 branes between A 5 brane and B 5 brane which is connected to C 5 brane. We have \widetilde{N}_{c2} between B 5 brane and C 5 brane which is connected to D 5 brane. There are $\widetilde{N}_{c1}/2$ between C 5 brane and D 5 brane which is connected to N_{f3} D6 branes by $3N_{f3}$ D4 branes. There are N_{f2} D6 branes to the right of N_{f3} D6 branes. Between C 5 brane and N_{f2} D6 branes we have $2N_{f2}$ D4 branes and between D 5 brane and N_{f2} D6 branes there are $2N_{f2}$ D4 branes also. Finally we have $2N_{f2}$ semiinfinite D4 branes ending on

N_{f2} D6 branes. Of course we have to add their mirrors in the above configuration.

The introduction of semiinfinite D4 branes is made in the same spirit of [11] and is necessary in order to be consistent with the result of geometric picture and of field theory which will be explained later. As in [26], the appearance of semiinfinite D4 branes in the brane configuration picture is the same as the fact that D6 branes must wrap on cycles between *pairs of singular points*. We will see that the results match if we introduce *exactly* this amount of semiinfinite D4 branes.

From the linking number conservation of D 5 brane, we can read it before the transition and after transition: $-N_{c3}/2 + N_{f3}/2 + N_{f2}/2 + N_{f1}/2 = \widetilde{N}_{c1}/2 - 3N_{f3} - 2N_{f2} + N_{f3}/2 + N_{f2}/2 - 2 - N_{f1}/2$. We obtain $\widetilde{N}_{c1} = 2(N_{f1} + 2N_{f2} + 3N_{f3}) - N_{c3} + 4$. Similarly the linking number of A 5 brane tells us that $\widetilde{N}_{c3} = 2(3N_{f1} + 2N_{f2} + N_{f3}) - N_{c1} + 4$ from the condition, $N_{c1}/2 - N_{f1}/2 - N_{f2}/2 - N_{f3}/2 = N_{f2}/2 - N_{f1}/2 + N_{f3}/2 + N_{f2} - \widetilde{N}_{c3}/2 + 3N_{f1} + 2$. Finally, we get $\widetilde{N}_{c2} = 2(N_{f1} + 2N_{f2} + N_{f3}) - N_{c2} - 2$ from the relation of the linking number of B 5 brane, $N_{c2} - N_{c1}/2 - N_{f2}/2 - N_{f3}/2 + N_{f1}/2 = N_{f2} - \widetilde{N}_{c2} + \widetilde{N}_{c3}/2 + N_{f2}/2 + N_{f3}/2 - N_{f1}/2 - 4$.

Now we go to the field theory results. The electric theory has the gauge group $SO(N_{c1}) \times Sp(N_{c2}) \times SO(N_{c3})$ with the following matter content of fields :

	$SO(N_{c1})$	$Sp(N_{c2})$	$SO(N_{c3})$	$U(1)_R$
$Q^{(1)}$	N_{c1}	1	1	$1 + \frac{1}{2N_{f1}}(N_{c2} - N_{c1} + 2)$
$Q^{(2)}$	1	2N_{c2}	1	$1 + \frac{1}{2N_{f2}}(\frac{N_{c1} + N_{c3}}{2} - 2 - 2N_{c2})$
$Q^{(3)}$	1	1	N_{c3}	$1 + \frac{1}{2N_{f3}}(2N_{c2} - N_{c3} + 2)$
X_1	N_{c1}	2N_{c2}	1	$\frac{1}{2}$
X_2	1	2N_{c2}	N_{c3}	$\frac{1}{2}$

The superpotential is given by

$$W = \frac{1}{2}\text{Tr}X_1^{4(k+1)} + \text{Tr}X_1^{2(k+1)}X_2^{2(k+1)} + \frac{1}{2}\text{Tr}X_2^{4(k+1)}.$$

To simplify the charges of these fields under different symmetries and to ease the writing of mesons of the theory, we took $k = 0$ in the above table. The form of the superpotential will determine the restrictions on the powers of X_1, X_2 , i.e., $X_1^5 = X_2^5 = 0$ which constrains the number of mesons which can be formed.

The mesons in the theory are given by

$$M_1^{(1)} = Q^{(1)}Q^{(1)}, M_2^{(1)} = Q^{(1)}X_1^2Q^{(1)}, M_3^{(1)} = Q^{(1)}X_1^4Q^{(1)},$$

$$M_1^{(2)} = Q^{(2)}Q^{(2)}, M_2^{(2)} = Q^{(2)}X_1^2Q^{(2)}, M_3^{(2)} = Q^{(2)}X_1^4Q^{(2)}, M_4^{(2)} = Q^{(2)}X_2^2Q^{(2)}, \\ M_0^{(3)} = Q^{(3)}Q^{(3)}, M_1^{(3)} = Q^{(3)}X_2^2Q^{(3)}, M_2^{(3)} = Q^{(3)}X_2^4Q^{(3)}.$$

The dual group is $SO(\widetilde{N}_{c1}) \times Sp(\widetilde{N}_{c2}) \times SO(\widetilde{N}_{c3})$ where $\widetilde{N}_{c1} = 2(3N_{f3} + 2N_{f2} + N_{f1}) - N_{c3} + 4$, $\widetilde{N}_{c2} = 4N_{f2} + 2N_{f1} + 2N_{f3} - N_{c2} - 2$, $\widetilde{N}_{c3} = 2(3N_{f1} + 2N_{f2} + N_{f3}) - N_{c1} + 4$. The content of the fields in the dual theory with their charges is as follows (the flavor group is the same in the dual theory as in the electric theory which is a common feature of $N = 1$ dualities):

	$SO(\widetilde{N}_{c1})$	$Sp(\widetilde{N}_{c2})$	$SO(\widetilde{N}_{c3})$	$U(1)_R$
$q^{(1)}$	$\widetilde{\mathbf{N}}_{\mathbf{c1}}$	$\mathbf{1}$	$\mathbf{1}$	$1 + \frac{1}{2N_{f1}}(\widetilde{N}_{c2} - \widetilde{N}_{c1} + 2)$
$q^{(2)}$	$\mathbf{1}$	$2\widetilde{\mathbf{N}}_{\mathbf{c2}}$	$\mathbf{1}$	$1 + \frac{1}{2N_{f2}}(\frac{\widetilde{N}_{c1} + \widetilde{N}_{c3}}{2} - 2 - 2\widetilde{N}_{c2})$
$q^{(3)}$	$\mathbf{1}$	$\mathbf{1}$	$\widetilde{\mathbf{N}}_{\mathbf{c3}}$	$1 + \frac{1}{2N_{f3}}(2\widetilde{N}_{c2} - \widetilde{N}_{c3} + 2)$
Y_1	$\widetilde{\mathbf{N}}_{\mathbf{c1}}$	$2\widetilde{\mathbf{N}}_{\mathbf{c2}}$	$\mathbf{1}$	$\frac{1}{2}$
Y_2	$\mathbf{1}$	$2\widetilde{\mathbf{N}}_{\mathbf{c2}}$	$\widetilde{\mathbf{N}}_{\mathbf{c3}}$	$\frac{1}{2}$

The fundamental quarks have reversed their global symmetries in the dual theory. There is a one-to-one map from the mesons of the original theory to the singlets of the dual theory. The singlets of the dual theory will enter in the superpotential which appear in the dual. Other checks of this duality can be performed. The t' Hooft anomaly matching conditions and the flow to other dualities by giving expectation values to different flavors are satisfied. The generalization to other values for k goes in the same way, the difference being the number of mesons which increases.

3.2 The addition of adjoint matter

Now we introduce adjoint matter. In the brane configuration picture, this corresponds to $(2k+1)$ NS5 branes with the same orientation as A connected by N_{c1} D4 branes with $(2k+1)$ NS5 branes with the same orientation as B. These are connected by N_{c2} D4 branes with $(2k+1)$ NS5 branes with the same orientation as C and the last ones are connected by N_{c3} D4 branes with $(2k+1)$ NS5 branes with the same orientation as D.

From the linking number conservation of D 5 brane, we can read it before the transition and after transition: $-N_{c3}/2 + (2k+1)N_{f3}/2 + (2k+1)N_{f2}/2 + (2k+1)N_{f1}/2 = \widetilde{N}_{c1}/2 - 3(2k+1)N_{f3} - 2(2k+1)N_{f2} + (2k+1)N_{f3}/2 + (2k+1)N_{f2}/2 - 2 - (2k+1)N_{f1}/2$. We obtain $\widetilde{N}_{c1} = 2(2k+1)(N_{f1} + 2N_{f2} + 3N_{f3}) - N_{c3} + 4$. Similarly the linking number

of A 5 brane tells us that $\widetilde{N}_{c3} = 2(2k+1)(3N_{f1} + 2N_{f2} + N_{f3}) - N_{c1} + 4$ from the condition $N_{c1}/2 - (2k+1)N_{f1}/2 - (2k+1)N_{f2}/2 - (2k+1)N_{f3}/2 = (2k+1)N_{f2}/2 - (2k+1)N_{f1}/2 + (2k+1)N_{f3}/2 + (2k+1)N_{f2} - \widetilde{N}_{c3}/2 + 3(2k+1)N_{f1} + 2$. Finally we get $\widetilde{N}_{c2} = 2(2k+1)(N_{f1} + 2N_{f2} + N_{f3}) - N_{c2} - 2$ from the relation of the linking number of B 5 brane $N_{c2} - N_{c1}/2 - (2k+1)N_{f2}/2 - (2k+1)N_{f3}/2 + (2k+1)N_{f1}/2 = (2k+1)N_{f2} - \widetilde{N}_{c2} + \widetilde{N}_{c3}/2 + (2k+1)N_{f2}/2 + (2k+1)N_{f3}/2 - (2k+1)N_{f1}/2 - 4$.

It is possible to obtain the duality for the 3-product with adjoint matter at the field theory level. For the same product gauge group, we introduce the fields $A_1(A_2)[A_3]$ in the antisymmetric(symmetric)[antisymmetric] adjoint representation of $SO(N_{c1})(Sp(N_{c2}))[SO(N_{c3})]$ besides the fields, X_1, X_2 that were in the previous subsection. Again we truncate the chiral ring by adding the superpotential:

$$W = \frac{1}{k+1} \text{Tr} A_i^{k+1} + \text{Tr} A_1 X_1^2 + \text{Tr} A_2 X_2^2 + \text{Tr} A_2 X_1^2 - \text{Tr} A_3 X_2^2, \quad i = 1, 2, 3$$

The conditions for supersymmetric vacuum reduce to the number of mesons. The dual group is $SO(2(2k+1)N_{f1} + 4(2k+1)N_{f2} + 6(2k+1)N_{f3} - N_{c3} + 4) \times Sp(2(2k+1)N_{f1} + 4(2k+1)N_{f2} + 2(2k+1)N_{f3} - N_{c2} - 2) \times SO(6(2k+1)N_{f1} + 4(2k+1)N_{f2} + 2(2k+1)N_{f3} - N_{c1} + 4)$. Again the flavors transform under different groups compared with the electric theory. The mesons of the electric theory become now singlets and will enter in the superpotential for the magnetic superpotential together with the dual mesons.

4 $SO(N_{c1}) \times Sp(N_{c2}) \times SO(N_{c3}) \times Sp(N_{c4})$

We want to see *new* dualities in the context of brane configurations which have not been studied before. Let us study 4-tuple product gauge group for the electric theory for the simplest case in this section where five NS5 branes have its single copy.

4.1 The theory with simplest superpotential

The gauge group is $SO(N_{c1}) \times Sp(N_{c2}) \times SO(N_{c3}) \times Sp(N_{c4})$ with $2N_{f1}(2N_{f2})[2N_{f3}]\{2N_{f4}\}$ flavors in the vector(fundamental)[vector]{fundamental} representation of respective $SO(N_{c1})(Sp(N_{c2}))[SO(N_{c3})]\{Sp(N_{c4})\}$ gauge group. We label them by letters A, B, C, D and E from left to right on the compact x^6 direction. The five NS5 branes are oriented at arbitrary angles in (x^4, x^5, x^8, x^9) directions. None of these NS5 branes are parallel.

There exist $N_{c1}/2$ D4 branes stretched between A 5 brane and B 5 brane which is connected to C 5 brane by N_{c2} D4 branes. The C 5 brane is connected to D 5 brane by $N_{c3}/2$ D4 branes. The D 5 brane is connected to E 5 brane by N_{c4} D4 branes. Between A(B)[C]{D} 5 brane and B(C)[D]{E} 5 brane we have $N_{f1}(N_{f2})[N_{f3}]\{N_{c4}\}$ D6 branes intersecting the $N_{c1}/2(N_{c2})[N_{c3}/2]\{N_{f4}\}$ D4 branes(plus their mirrors).

In order to go to magnetic dual theory, by moving all the N_{f1} D6 branes to the left of all NS5 branes, each D6 brane has four D4 branes on its left after transition and by moving all N_{f4} D6 branes to the right, each D6 brane has four D4 branes on its right. First move B 5 brane to the right of A 5 brane. Two D4 branes must appear because we have SO gauge group. We move C 5 brane to the right of A 5 brane. When C 5 brane passes A 5 brane, two D4 branes disappear between A 5 brane and C 5 brane because we have an Sp gauge group, so we have two D4 branes between C 5 brane and B 5 brane and no extra D4 branes between A 5 brane and C 5 brane. Now move D 5 brane to the right of A 5 brane. When D 5 brane passes A 5 brane there are two new D4 branes between them because of the SO gauge group. Similarly we move C 5 brane to the right of B 5 brane there are two new D4 branes but the previous two D4 branes are changing the orientation so they annihilate the two new D4 branes which leads eventually to no D4 branes created or annihilated. The gauge group in the magnetic theory is then given by $SO(\tilde{N}_{c1}) \times Sp(\tilde{N}_{c2}) \times SO(\tilde{N}_{c3}) \times Sp(\tilde{N}_{c4})$ with inverted order of the corresponding flavors. As we did before, the values for $\tilde{N}_i(i = 1, 2, 3, 4)$ are calculated by linking number conservations and we obtain the following values: $\tilde{N}_{c1} = 2(N_{f1} + 2N_{f2} + 3N_{f3} + 4N_{f4}) - 2N_{c4}$, $\tilde{N}_{c2} = 2N_{f1} + 4N_{f2} + 6N_{f3} + 3N_{f4} - N_{c3}/2$, $\tilde{N}_{c3} = 2(3N_{f1} + 6N_{f2} + 4N_{f3} + 2N_{f4}) - 2N_{c2}$, $\tilde{N}_{c4} = 4N_{f1} + 3N_{f2} + 2N_{f3} + N_{f4} - N_{c1}/2$.

5 Generalization to higher product gauge groups

We have seen that a number of $N = 1$ supersymmetric field theory dualities were obtained in terms of brane configuration and geometric realization of wrapping D6 branes around 3-cycles of CY threefold in type IIA string theory. Our construction also gives rise to extra D6 branes wrapping around the cycles. It would be interesting to study this transition further. The above construction can be generalized to any product of gauge groups, but we have to put them in alternating order, i.e., $\cdots SO(N_{c1}) \times Sp(N_{c2}) \times SO(N_{c3}) \times Sp(N_{c4}) \times \cdots$ (by changing the overall sign of Ω^2 we can start with a Sp gauge group from left to right).

In order to obtain a generalization to any value of n , one has to take $(2n - 1)$ singular points and to move them from left to right with respect to a reference point. We always have to be able to extend the flavor cycles to infinity. So we always push a_2 to the far

right and c_2, d_2, \dots to the far left. In order to obtain the result of field theory, we have to twist the fibration in order to move the D 6-branes which do not contribute to the gauge group to the desired direction. This is equivalent to introducing the semi-infinite D 4-branes in brane configuration method. Our result again agrees with the one of [11].

For a product of more than two gauge groups, there are two cases:

- 1) when there is an even number of gauge groups in the product, i.e., $\prod_{i=1} SO(N_{ci}) \times Sp(N_{c(i+1)})$ where $i = 2j - 1$ and $j = 1, 2, \dots$ the effects of SO and Sp projections will cancel each other so in the overall picture of the dual, no D4 branes appear or disappear. The result is similar to the one obtained by [11] with the modifications that are to be done when one considers non-orientable string theory. Taking their results, we simply modify \tilde{N}_c to $2\tilde{N}_c$ and N_c to $2N_c$ anytime we talk about the Sp gauge groups, obtaining the dual theory for the alternating product of SO and Sp gauge groups. The argument that we use for the product of two gauge groups applies also here. So one has to be careful when changing the positions of two NS5 branes connected by supplementary D4 branes or having a deficit of D4 branes between them.
- 2) when there is an odd number of gauge groups in the product, i.e., $(\prod_{i=1} SO(N_{ci}) \times Sp(N_{c(i+1)})) \times SO(N_c)$ where $i = 2j - 1$ and $j = 1, 2, \dots$, we need to create or to annihilate D4 branes in the overall picture(or to put D4 branes or anti D4 branes on the top of the orientifold in order to make a smooth transition).

6 Conclusions

Originally, the dualities for $N=1$ supersymmetric theories were considered between theories which had the same quantum chiral ring and moduli space of vacua as a function of possible deformations determined by the expectation values of the scalar fields of both theories. It is very difficult to find the dual theory because nobody knows a specific recipe for obtaining it. The models were considered separately and only some examples which enter in a general scheme have been obtained. By embedding the problem in string theory duality is translated in brane mechanics. The classical moduli space for both theories are embedded in a single moduli space of string vacua and this makes the relation between the two moduli space manifest. The quarks, antiquarks and mesons have a natural interpretation as different strings forced to end on different D branes and as oscillations of the specific D4 branes. The superpotential for the specific models considered so far appears from their $N = 2$ origin after rotating NS5 branes.

In the geometrical picture, the relation between the moduli spaces of both theories

becomes again manifest. In this approach, we just go between different points of the moduli space of Calabi-Yau manifolds. By wrapping D6 branes on the vanishing three cycles on the doubly elliptic fibered Calabi-Yau manifold, we realize $N = 1$ dualities in four dimensions for various gauge groups. Via T-duality, we connect these geometric configurations to the brane configurations. In brane configuration picture, semi-infinite branes are to be introduced and we always have to check that the result is identical with the one obtained in geometric picture and by field theory methods.

Many results have been obtained before for different gauge groups and several matter representations. In the present work we generalized the work of [14, 26] to the case of product gauge groups between SO and Sp gauge groups. The main ideas, in the geometric picture, were the appearance of supplementary D6 branes when the positions of two singular points are inverted and the fact that in magnetic dual theory the flavor groups are given by D6 branes wrapped on cycles between *pairs* of singular points. In brane configuration picture the last condition was translated in the appearance of semi-infinite D4 branes as in [11].

All these $N = 1$ configurations are obtained from $N = 2$ configurations by rotating the NS5 branes. Considering the fact that in $N = 2$ theories we cannot have $SO(N_c)$ gauge group with $SO(N_f)$ flavor group, this implies that, by gauging the flavor group, we cannot have $SO(N_c) \times SO(N_f)$ gauge group. Using the arguments of [10, 13], this can be understood by using the fact that the charge of the O4 plane is connected with the sign of Ω^2 so for each NS5 brane, we must have SO projection on one side and Sp projection on the other side. Therefore this brane configuration seems not to be suitable for describing the gauge theory with $SO(N_{c1}) \times SO(N_{c2})$ gauge group or $Sp(N_{c1}) \times Sp(N_{c2})$ gauge group. As we know at the field theory level these dual relations have been found already in the paper of [46]. It would be interesting to study how these dualities arise from brane configuration or geometric configuration. Another very interesting case is the gauge theory with $SO(N_{c1}) \times SU(N_{c2})$ where the orientifold should act only on one part of the configuration. It is again clear that the presently known brane configuration cannot describe this model. Matter in the spinor representation would allow us to obtain other very interesting model, especially the duality between chiral theories with SO group and non-chiral theories with SU group. All this possible ideas are urging us to find a new configuration preserving $N = 1$ in $d = 4$ dimensions.

In the beautiful lectures of Townsend given at Cargese School 1997, there appear to be few configurations which preserves 1/8 of supersymmetries. In order to obtain an effective four-dimensional theories, some D branes should have compact directions such that the world seen by an observer who lives on them is a four-dimensional field theory. In all the cases considered so far, D4 branes with one compact directions were considered. It seems that some possibilities for other interesting configurations would be to consider D4 branes which are compact in more than one direction, like a D5

brane ending on D7 brane and with two compact directions. It should be extremely interesting to obtain new brane configurations which preserve 1/8 of supersymmetry and simultaneously which cannot be obtain from the configurations which preserve 1/4 of supersymmetry.

Therefore we expect many new developments in this line of research which will give us many new insights into field theory dualities and also string theories with a better understanding of brane configurations. Developments in both theories have a common goal of obtaining information about the strong coupled sectors of both theories.

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7 Appendix

We give here a short explanation for the new phenomenon occurring in the transition from the electric to magnetic theory in the OV approach. Let us discuss for the orientable case, i.e., with SU groups. Consider a NS5 brane denoted by A in (12345) directions, another NS5 brane denoted by B in (12389) directions and N_f D4 branes between them in (1236) directions. We do not have any D6 branes. We want to move the B 5 brane from the left to the right of the A 5 brane. Because we do not have any D6 brane, we cannot make the transition to the Higgs phase. So, in order to preserve supersymmetry, we are not allowed to move the B 5 brane in the x^7 direction. In order to pass it to the left of the A 5 brane we have to pass the NS5 branes one over the other. But the D4 branes are bound to the 5 brane B and they will pass through the stationary NS5 brane. In order to see better the phenomenon, we consider that the NS5 branes are very closed to each other, so that the extension of the D4 branes in the x^6 direction is negligible. Then we make a succession of dualities :

- 1) T-duality on (123) direction; the D4 brane becomes D1 brane(6) but x^6 is negligible so it can be consider to be just a point stuck on the NS branes.
- 2) S - duality so we have D5 brane(12345) and D5 brane(12389).

3) T-duality on x^7 direction so we obtain D6 brane(123457) and D6 brane(123789).

So we have one D6 brane (123457) passing through a D6 brane (123789). If the (12367) space were a 5 torus, then our problem would reduce to the one of [4]. In that case, formula (6) of [4] tells us that the net charge inflow is k where k is the instanton number. Here the instanton number is given by the charge of the D4 branes which became points after the first three T-dualities. So the net charge inflow is just N_f . This tells us that there is a fundamental string in the x^6 direction which is created after the NS5 branes are passing through each other. If we go backwards with the three steps of dualities, we obtain from the F1 string(6) one D6 brane(1236). So, if the A and B NS5 branes are passing each other, there are N_f D4 branes that are created between the two NS5 branes. The argument is not changed if we take the volume of the 5 torus to be very big and we go to the noncompact limit. This is the phenomenon that occurs in the OV approach.

In the non-orientable case everything goes the same, but we have to count always the number of D4 branes as the sum of physical D4 branes plus their mirrors.

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